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YET ANOTHER POINCARÉ'S POLYHEDRON THEOREM

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Abstract. Poincaré's Polyhedron Theorem is a widely known valuable tool in constructing manifolds endowed with a prescribed geometric structure. It is one of the few criteria providing discreteness of groups of isometries. This work contains a version of Poincaré's Polyhedron Theorem that is applicable to constructing fibre bundles over surfaces and also suits geometries of nonconstant curvature. Most conditions of the theorem, being as local as possible, are easy to verify in practice.

1. Introduction

1.1. It is frequently important to decide if a given subgroup G of a Lie group is discrete. For instance, such a necessity appears while constructing manifolds endowed with a prescribed geometric structure. Typically, the group G is related to some geometrical configuration: it acts on a simply-connected homogeneous space M and is generated by isometries that identify given codimension 1 subspaces called faces. These faces may bound a (fundamental) polyhedron P in M into which the quotient M/G can be 'cut and unfolded.' Thus, we expect certain pairs of faces to be identified by the generators of G, called face-pairing isometries, in such a way that M/G results from the identifications and the space M is tessellated by the copies of P. This can be reversed: starting with a polyhedron that tessellates M, we get the discrete group generated by the face-pairing isometries. We have just briefly described the general settings surrounding Poincaré's Polyhedron Theorem (PPT). The theorem has a long history and plenty of versions; the interested reader may consult, for instance, [EPe] and the references therein.

The main step in verifying the tessellation of M is usually the study of tessellation around the codimension 2 faces of P, called edges. This leads to the concept of a geometric cycle of edges: given an edge e, its geometric cycle is a cyclic sequence of edges related by face-pairing isometries such that the corresponding copies of P are expected to tessellate M around e. Dealing with geodesic polygons in the hyperbolic plane, Henri Poincaré realized that in order to obtain the tessellation of M it suffices to require that the sum of the angles of the polygon along every geometric cycle equals 2π (for simplicity, we do not deal here with ideal cycles). Later, he extended this idea to the case of constant curvature hyperbolic 3-space.

1.2. Most versions of PPT come from constant curvature (or even plane) geometries, where convexity arguments play an important role typically suited to polyhedra with constant angles between (totally geodesic) faces along common edges. In general, such an approach is inapplicable to nonconstant curvature geometries, say, to complex hyperbolic geometry. The usual requirements like 'adjacent polyhedra intersect in an expected way' are difficult to check. Therefore, we look for a version where the conditions for tessellation are as local as possible and provide global properties just *a posteriori.* The strategy is to impose some requirements of infinitesimal nature which can be verified in practice and then obtain an infinitesimal tessellation that can be 'integrated' with the help of suitable local conditions expressing a good behaviour of the faces. Note that while constructing manifolds we are used to having an explicitly given set of face-pairing isometries. We therefore treat the relation between the face-pairing isometries involved in a cycle of edges as being easily verifiable: at worst, we need to multiply a few matrices.

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1.3. We show that the tessellation of a metric neighbourhood (see Tessellation Condition 2.1) is sufficient for discreteness (Proposition 2.2). This condition seems to be more useful than the well-known completeness requirement. We think it can be particularly relevant in dimension >2 if one also has to deal with the 'parabolic cycles.' Conditions similar to Tessellation Condition 2.1 have already appeared in the literature (see, for instance, [Ale] and [Bea]).

The Poincaré angle condition can be weakened to the following total angle condition. Pick a point p in an edge e. Being subsequently applied to p, the face-pairing isometries involved in the geometric cycle of e provide a point in each edge in the cycle. We measure the interior angles between faces at these points. The total angle at p, i.e., the sum of such angles, is a multiple of 2π because of the cycle relation. *Total angle condition* means that, for every cycle, the total angle at a single point equals 2π. Such a condition can be quite handy in particular cases: sometimes the verification happens to be very simple at geometrically distinguished points.

In Theorem 3.5, we deal with polyhedra that possess no faces of codimension >2. In this case, total angle condition essentially ensures the tessellation of a topological neighbourhood of the polyhedron. Requiring in addition the metric separability of faces (including that of faces sharing a common edge; see Strong Simplicity IV and Condition (3) in Theorem 3.5) allows us to integrate the topological tessellation into the tessellation of a metric neighbourhood of the polyhedron. In Final Remarks 4.3, we explain how one can in principle use the ideas of [Ale] and the current paper to obtain a more general form of the theorem with no restriction on codimension of faces. As it stands, Theorem 3.5 is well adapted to the construction of fibre bundles over surfaces (in order of importance, such manifolds are probably the first after the compact ones); for example, it applies directly to constructing complex hyperbolic disc bundles in [AGG].

A serious defect of our version is the global requirement of simplicity, i.e., the requirement that the faces intersect as expected and thus bound the polyhedron itself. In complex hyperbolic geometry, for example, it is already difficult to check the simplicity of a polyhedron with bisectors taken as faces. Hence, it seems that one should obtain an even more local PPT which makes the verification of simplicity unnecessary. This would finally 'disassemble' the polyhedron, taking away the arbitrary choice involved in PPT.

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2. Preliminaries

This is essentially standard material (see, for instance, [Bea, §9.8, p. 242]).

Let M be a locally path-connected, connected, and simply-connected metric space. Denote by $B(x,\varepsilon)$ the open ball of radius $\varepsilon>0$ centred at x and let $N(X,\varepsilon):=\bigcup\limits_{x\in X}B(x,\varepsilon)$ for $X\subset M$. We regard a *polyhedron* in M as being a closed, locally path-connected, and connected subspace $P\subset M$ such that

• P is the closure of its nonempty interior: $\mathring P\ne\varnothing$ and $P=\operatorname{Cl}\mathring P$;

• the nonempty boundary of P is decomposed into the union of nonempty subsets $s\in S$ called *faces*: $\partial P:=P\setminus\mathring P=\bigcup\limits_{s\in S}s$.

A *face-pairing* of a polyhedron P is an involution $\overline{}:S\to S$ and a family of isometries $I_s\in\operatorname{Isom}M$ satisfying $I_ss=\overline s$ and $I_{\overline s}=I_s^{-1}$ for every face $s\in S$.

Let P be a polyhedron with a given face-pairing and let G denote the group generated by the face-pairing isometries. We introduce a relation in $G\times P$ by putting $(g,x)\sim(h,y)$ exactly when $x\in s$ for some $s \in S$, $I_s x = y$, and $h^{-1}g = I_s$. Taking the closure of this symmetric relation with respect to transitivity (and reflexivity), we obtain an equivalence relation also denoted by $\sim$. Let $J := G \times P/\sim$ and let $[g, x]$ denote the class of (g, x) in J. Consider the discrete topology on G and equip P, $G \times P$, and J with their natural topologies. We have a commutative diagram of continuous G-maps $\psi(g, x) := gx$, $\pi(g, x) := [g, x]$, and $\varphi[g, x] := gx$. (Actions of G by homeomorphisms are defined by $h(g, x) := (hg, x)$ and $h[g, x] := [hg, x]$.) Let

$$\begin{tikzcd} G \times P \arrow[r, "\pi"] \arrow[dr, "\psi"'] & J \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

$$[P] := \big\{[1, x] \mid x \in P\big\} \quad \text{and} \quad [\mathring{P}] := \big\{[1, x] \mid x \in \mathring{P}\big\}.$$

Clearly, $J = \bigcup\limits_{g\in G} g[P]$ and $g_1[\mathring{P}] \cap g_2[\mathring{P}] \neq \varnothing$ implies $g_1 = g_2$. In other words, $[P]$ is a *fundamental region* for the action of G on J.

We assume that $\pi^{-1}[1, x]$ is **finite** for every $x \in \partial P$, hence, for every $x \in P$. Let $x \in P$. Then $\pi^{-1}[1, x] = \big\{(g_1, x_1), \dots, (g_n, x_n)\big\}$ for some $g_j \in G$ and $x_j \in P$. The polyhedra $g_j P$ are the *formal neighbours* of P at x. For $\delta > 0$, define

$$N_{x_j,\delta} := \big\{y \in P \mid d(y, x_j) < \delta\big\} \subset P, \qquad N_{x,\delta} := \bigcup_{j=1}^{n} (g_j, N_{x_j,\delta}) \subset G \times P, \qquad W_{x,\delta} := \pi N_{x,\delta} \subset J,$$

where $d(\cdot, \cdot)$ stands for the distance function on M. Using this notation, we state the

2.1. Tessellation Condition. A polyhedron P with a given face-pairing satisfies Tessellation Condition if

• for every $x \in P$, there exists some $\delta(x) > 0$ such that $\pi^{-1}(W_{x,\delta}) = N_{x,\delta}$ and $\varphi W_{x,\delta} = B(x, \delta)$ for all $0 < \delta \leq \delta(x)$;

• some open metric neighbourhood N of P in M is tessellated; this means that $N(P, \varepsilon) \subset N$ for some $\varepsilon > 0$ and that there exists a function $f : P \to \mathbb{R}$ taking positive values such that $\varphi : W_{P,f} \to N$ is bijective, where $W_{P,f} := \bigcup\limits_{x\in P} W_{x,f(x)}$.

2.2. Proposition. *Tessellation Condition* 2.1 *implies that φ is a homeomorphism. In other words, the polyhedron P is a fundamental region for the action of G on M.*

Proof. Straightforward arguments show that J is Hausdorff and path-connected, that the family $\big\{gW_{x,\delta} \mid g \in G,\ x \in P,\ 0 < \delta \leq \delta(x)\big\}$ is a base of the topology on J, and that φ is a local homeomorphism. Clearly, $\varphi : gW_{P,f} \to gN$ is a homeomorphism for all $g \in G$. As M is simply-connected, it suffices to show that φ is a regular covering.

Since φ is open, φJ is open in M. Let $x \in \operatorname{Cl}(\varphi J)$. Then $B(x, \varepsilon) \cap gP \neq \varnothing$ for some $g \in G$. It follows that $x \in N(gP, \varepsilon) \subset gN = \varphi(gW_{P,f}) \subset \varphi J$. Hence, φJ is closed in M. Since M is connected, φ is surjective.

Take $x \in M$. Define
$$G_x := \big\{g \in G \mid U_x \cap gP \neq \varnothing\big\},$$
where $U_x \subset B(x, \frac{1}{2}\varepsilon)$ is a path-connected open neighbourhood of x. For every $g \in G_x$, let

$$W_g := \varphi^{-1}(U_x) \cap gW_{P,f}.$$

Since $U_x \cap gP \neq \varnothing$ implies that $U_x \subset B(x, \frac{1}{2}\varepsilon) \subset N(gP, \varepsilon) \subset gN$, we conclude that $\varphi : W_g \to U_x$ is a homeomorphism. Moreover,
$$\varphi^{-1}(U_x) = \bigcup_{g\in G_x} W_g.$$

It remains to show that the distinct W_g's are disjoint. Suppose that $W_{g_1}\cap W_{g_2}\neq\varnothing$ for some $g_1,g_2\in G_x$. The projection $W_{g_1}\times W_{g_2}\to W_{g_1}$ induces a homeomorphism between

$$X:=\big\{(x_1,x_2)\in W_{g_1}\times W_{g_2}\mid \varphi x_1=\varphi x_2\big\}$$

and W_{g_1}. The diagonal

$$\Delta_{W_{g_1}\cap W_{g_2}}=\Delta_J\cap(W_{g_1}\times W_{g_2})\subset X$$

is closed in X since J is Hausdorff. Therefore, the image $W_{g_1}\cap W_{g_2}$ of $\Delta_{W_{g_1}\cap W_{g_2}}$ is closed in W_{g_1}. Since W_{g_1} is connected, we obtain $W_{g_1}=W_{g_2}$ ■

3. A Plane-like Poincaré's Polyhedron Theorem

In what follows, M is a connected, oriented, and simply-connected Riemannian manifold. We regard a *cornerless polyhedron* $P\subset M$ with a face-pairing as a subspace satisfying the conditions stated in the beginning of the previous section as well as those below.

I. The faces of P are topologically closed, oriented smooth connected submanifolds of codimension 1 in M with (possibly empty) boundary. Each face s of P is oriented so that normal vectors to $s\setminus\partial s$ point towards the interior of P.

II. The boundary of every face $s\in S$ is a disjoint union $\partial s=\bigsqcup\limits_{e\in E_s}e$ of nonempty connected *edges.* ($E_s=\varnothing$ is allowed.) We write $e\diamond s$ or $s\diamond e$ if $e\in E_s$. Clearly, $e\diamond s$ implies $\overline{s}\diamond I_se$.

III. P has a finite number of faces and edges. Each edge e belongs to exactly two distinct faces s_1 and s_2. In symbols: $s_1\diamond e\diamond s_2$.

IV (Strong Simplicity)**.** The intersection of two distinct faces is contained in the boundary of both faces and is a (possibly empty) union of edges. The distances between:
two distinct edges,
two distinct faces that do not share an edge,
a face and an edge not contained in it
are all greater than some $d>0$.

3.1. Start with $\overline{s}_0\diamond e\diamond s_1$. Applying I_{s_1} to s_1 and e, we obtain $\overline{s}_1\diamond I_{s_1}e\diamond s_2$. Applying I_{s_2} to s_2 and $I_{s_1}e$, we obtain $\overline{s}_2\diamond I_{s_2}I_{s_1}e\diamond s_3$, and so on. (Of course, by III, we eventually arrive back at $\overline{s}_0\diamond e\diamond s_1$.)

A cyclic sequence

$$\xrightarrow{I_{s_n}}\overline{s}_n=\overline{s}_0\diamond e\diamond s_1\xrightarrow{I_{s_1}}\overline{s}_1\diamond I_{s_1}e\diamond s_2\xrightarrow{I_{s_2}}\overline{s}_2\diamond I_{s_2}I_{s_1}e\diamond s_3\xrightarrow{I_{s_3}}\dots\xrightarrow{I_{s_{n-1}}}\overline{s}_{n-1}\diamond I_{s_{n-1}}\cdots I_{s_1}e\diamond s_n\xrightarrow{I_{s_n}},$$

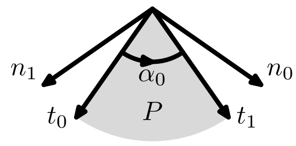
where each term is obtained from the previous one by the above rule, is called a *cycle of edges.* The number n is the length of the cycle and the isometry $I:=I_{s_n}\cdots I_{s_1}$ will be referred to as the *cycle isometry.* A cycle can be read backwards, i.e., in opposite *orientation,* which inverts its isometry. If the cycle isometry is the identity and if the cycle is the shortest one with this property, then the cycle is said to be *geometric* (see also Remark 3.4.) Clearly, every cycle is a multiple of a shortest, combinatorial one. Note that, in a geometric cycle, an edge may occur several times (this does not happen in a combinatorial cycle).

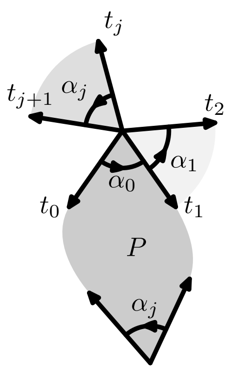
Assume that we are given a family of disjoint geometric cycles that contains every edge of P. Fixing a term (say) $\overline{s}_0\diamond e\diamond s_1$ in some oriented cycle of the family, define $I_j:=I_{s_j}\cdots I_{s_1}$ for all $j=0,1,\dots,n$ (we usually consider j modulo n) so that the cycle takes the form

$$\xrightarrow{I_{s_n}}\overline{s}_n=\overline{s}_0\diamond I_0e\diamond s_1\xrightarrow{I_{s_1}}\overline{s}_1\diamond I_1e\diamond s_2\xrightarrow{I_{s_2}}\overline{s}_2\diamond I_2e\diamond s_3\xrightarrow{I_{s_3}}\dots\xrightarrow{I_{s_{n-1}}}\overline{s}_{n-1}\diamond I_{n-1}e\diamond s_n\xrightarrow{I_{s_n}}.$$

3.2. We can describe all formal neighbours of P at a point $x \in \partial P$. If x does not belong to any edge, then there is a unique face s containing x, $\pi^{-1}[1,x] = \big\{(1,x),(I_{\overline{s}}, I_s x)\big\}$, and the only formal neighbours of P at x are P and $I_{\overline{s}}P$. If x belongs to an edge $\overline{s}_0 \diamond e \diamond s_1$, then $\pi^{-1}[1,x] = \big\{(I_j^{-1}, I_j x) \mid j = 0,1,\dots,n-1\big\}$ and the $I_j^{-1}P$ are the formal neighbours of P at x. Indeed, suppose that $(I_j^{-1}, I_j x) \sim (h,y)$. This means that $I_j x \in s'$, $I_{s'} I_j x = y$, and $h^{-1} I_j^{-1} = I_{s'}$ for some $s' \in S$. In particular, $I_j e$ and s' intersect. It follows from IV that $I_j e \diamond s'$. Hence, either $s' = \overline{s}_j$ or $s' = s_{j+1}$. Therefore, either $(h,y) = (I_{j-1}^{-1}, I_{j-1}x)$ or $(h,y) = (I_{j+1}^{-1}, I_{j+1}x)$. It remains to observe that the I_j, $j = 0,1,\dots,n-1$, are all distinct because we could otherwise take a shorter cycle whose isometry would be the identity.

3.3. Pick a point x in some edge $\overline{s}_0 \diamond e \diamond s_1$ of an oriented cycle. Let $\mathrm{N}_x\, e := (\mathrm{T}_x\, e)^\perp$ and let n_0, n_1 denote, respectively, the unit normal vectors to $\overline{s}_0, s_1$ at x that point towards the interior of P. Let $t_0 \in \mathrm{T}_x\, \overline{s}_0 \cap \mathrm{N}_x\, e$ and $t_1 \in \mathrm{T}_x\, s_1 \cap \mathrm{N}_x\, e$ stand for the unit vectors that point respectively towards the interiors of $\overline{s}_0$ and s_1. The basis t_0, n_0 orients $\mathrm{N}_x\, e$. This orientation corresponds to the orientation of the cycle. The oriented *interior angle* α_0 from $\overline{s}_0$ to s_1 at x is the angle from t_0 to t_1, which takes values in $[0, 2\pi]$. We define similarly the interior angle α_j from $\overline{s}_j$ to s_{j+1} at $I_j x$. The sum $\sum_{j=0}^{n-1} \alpha_j$ is the *total interior angle* of the cycle at x. It is easy to see that altering the orientation of the cycle alters the orientation of the corresponding $\mathrm{N}_x\, e$ and keeps the same values of the α_j's.





Suppose that the face-pairing isometries *send interior into exterior*. By definition, this means that $I_s n_s = -n_{\overline{s}}$ for every face $s \in S$, where n_s stands for the unit normal vector to s at some $x \in s$. This property implies the following. Take a point x in some edge $\overline{s}_0 \diamond e \diamond s_1$ of an oriented geometric cycle. Let $t_1 \in \mathrm{T}_x\, s_1 \cap \mathrm{N}_x\, e$ be the unit vector that points towards the interior of $I_1^{-1}\overline{s}_1 = s_1$ and let $t_2 \in \mathrm{T}_x\, I_1^{-1} s_2 \cap \mathrm{N}_x\, e$ be the unit vector that points towards the interior of $I_1^{-1} s_2$. Then the oriented angle from t_1 to t_2 equals α_1. In the same way, denoting by $t_j \in \mathrm{T}_x\, I_j^{-1}\overline{s}_j \cap \mathrm{N}_x\, e$ the unit vector that points towards the interior of $I_j^{-1}\overline{s}_j$, we can see that the oriented angle from t_j to t_{j+1} equals α_j. This implies immediately that $\sum_{j=0}^{n-1} \alpha_j \equiv 0 \mod 2\pi$. In particular, the total interior angle of a geometric cycle is **constant**: it does not depend on the choice of $x \in e$.

Obviously, the distinct formal neighbours of P at a point in an edge overlap when the total interior angle of a cycle is different from 2π. In the terms of Proposition 2.2, this corresponds to a ramification of φ.

3.4. Remark. For some geometries, the nature of edges allows to (formally) weaken the condition that the cycle isometry is the identity. This happens in the case when every isometry I that fixes pointwise some edge e is completely determined by the rotation angle about some $x \in e$, that is, by the image $In \in \mathrm{N}_x\, e$ of some $0 \neq n \in \mathrm{N}_x\, e$. In this case, it suffices to require only that $I|_e = 1_e$ and that the total interior angle at x vanishes modulo 2π.

3.5. Theorem. *Let P be a cornerless polyhedron with a face-pairing providing a family of geometric cycles that contains every edge of P. Suppose that*

(1) *the face-pairing isometries send interior into exterior*;

(2) *the total interior angle equals 2π at some point of an edge for every cycle of the family*;

(3) *for every two distinct faces s, s' such that $s \cap s' \neq \varnothing$ and for every $\vartheta > 0$, there exists $\varepsilon = \varepsilon(s,s',\vartheta) > 0$ such that $s' \cap N(s,\varepsilon) \subset \bigcup\limits_{s \diamond e \diamond s'} N(e,\vartheta)$.*

Then Tessellation Condition 2.1 *is satisfied.*

Proof. In what follows, we denote $\widetilde{X} := X \cap P$ for $X \subset M$.

First step. Using Conditions (1–2), we will integrate (employing IV) an infinitesimal tessellation into a topological one. So, we will show that there exists a sufficiently small tessellated open ball centred at x for every $x \in P$. In other words, for every $x \in P$, we will find some $\delta(x) > 0$ such that the first part of Tessellation Condition 2.1 is valid and, additionally, $\varphi : W_{x,\delta} \to B(x,\delta)$ is injective for all $0 < \delta \leq \delta(x)$. We distinguish the cases $x \in s \setminus \partial s$ for some $s \in S$ and $x \in e$ for some edge e.

• In the first case, choose $\delta_1 > 0$ such that $B(x,\delta_1)$ does not intersect the edges of s and such that $B(x,\delta_1) \cap \partial P = B(x,\delta_1) \cap s$. Choose $\delta_2 > 0$ analogously with respect to $I_s x \in \overline{s}$. Let

$$N_{x,\delta} := \big(1, \widetilde{B}(x,\delta)\big) \bigsqcup \big(I_{\overline{s}}, \widetilde{B}(I_s x,\delta)\big), \qquad W_{x,\delta} := \pi N_{x,\delta},$$

where $0 < \delta \leq \delta(x) := \min(\delta_1,\delta_2)$. Clearly, $\pi^{-1} W_{x,\delta} = N_{x,\delta}$. We need to show that $\varphi : W_{x,\delta} \to B(x,\delta)$ is a bijection.

Note that $s \cap B(x,\delta) \subset \widetilde{B}(x,\delta) \cap I_{\overline{s}}\widetilde{B}(I_s x,\delta)$. Also, $\widetilde{B}(x,\delta) \neq I_{\overline{s}}\widetilde{B}(I_s x,\delta)$ by Condition (1). Pick a point $q_0 \in \widetilde{B}(x,\delta) \setminus I_{\overline{s}}\widetilde{B}(I_s x,\delta)$ such that $q_0 \notin s$. Due to the fact that $\delta \leq \delta(x)$, a smooth oriented curve $\gamma \subset B(x,\delta)$ connecting q_0 and $q \in B(x,\delta) \setminus s$ can intersect ∂P and $\partial I_{\overline{s}} P$ only along $(s \setminus \partial s) \cap B(x,\delta)$. We can assume that such intersections are transverse. According to (1), when intersecting s, the curve γ leaves $\widetilde{B}(x,\delta)$ and enters $I_{\overline{s}}\widetilde{B}(I_s x,\delta)$ or *vice-versa.* Hence, q belongs to exactly one of $\widetilde{B}(x,\delta)$ and $I_{\overline{s}}\widetilde{B}(I_s x,\delta)$. The result then follows.

• The second case is similar. Let the I_j's be related to the geometric cycle containing e. Choose $\delta_j > 0$ such that $B(I_j x,\delta_j)$ does not intersect any edge of $\overline{s}_j$ or s_{j+1} except $I_j e$ and such that $B(I_j x,\delta_j) \cap \partial P = \big(B(I_j x,\delta_j) \cap \overline{s}_j\big) \cup \big(B(I_j x,\delta_j) \cap s_{j+1}\big)$. Let

$$N_{x,\delta} := \bigsqcup_j \big(I_j^{-1}, \widetilde{B}(I_j x,\delta)\big), \qquad W_{x,\delta} := \pi N_{x,\delta},$$

where $0 < \delta \leq \delta(x) := \min \delta_j$. The description 3.2 of formal neighbours implies that $\pi^{-1}(W_{x,\delta}) = N_{x,\delta}$. We have

$$I_j^{-1} s_{j+1} \cap B(x,\delta) \subset I_j^{-1}\widetilde{B}(I_j x,\delta) \cap I_{j+1}^{-1}\widetilde{B}(I_{j+1} x,\delta).$$

Let $F := \bigcup\limits_j I_j^{-1} s_{j+1} \cap B(x,\delta)$ and let $q_0 \in \widetilde{B}(x,\delta) \setminus F$. Since $\delta \leq \delta(x)$, a smooth oriented $\gamma \subset B(x,\delta)$ connecting q_0 and $q \in B(x,\delta) \setminus F$ may intersect $\bigcup\limits_j \partial I_j^{-1} P$ only along F. We assume that γ does not intersect e and is transverse to F. Condition (1) implies that, when intersecting $I_j^{-1} s_{j+1}$, the curve γ leaves $I_j^{-1}\widetilde{B}(I_j x,\delta)$ and enters $I_{j+1}^{-1}\widetilde{B}(I_{j+1} x,\delta)$ or *vice-versa.* Hence, $\varphi : W_{x,\delta} \to B(x,\delta)$ is surjective. Following the discussion 3.3 concerning the total angle of the cycle at x, we consider the closed sectors $T_j \subset \mathrm{N}_x\, e$ containing the oriented interior angle of $I_j^{-1} P$ at x. Conditions (1) and (2) imply that $\bigcup\limits_j T_j = N_x e$ and $\mathring{T}_{j_1} \cap \mathring{T}_{j_2} = \varnothing$ if $j_1 \not\equiv j_2 \mod n$. Hence, distinct formal neighbours $I_j^{-1} P$ cannot be equal.

Suppose that $\varphi : W_{x,\delta} \to B(x,\delta)$ is not injective at some $q \in B(x,\delta)$. It follows from the description 3.2 of formal neighbours that $q \notin F$. Pick a point q_0 living in exactly one of the $I_j^{-1}\widetilde{B}(I_j x,\delta) \setminus F$ and connect q_0 and q by a smooth oriented curve $\gamma \subset B(x,\delta)$ that does not intersect e and is transverse to F. By the properties of δ and the above 'leaves-and-enters' argument, we arrive at a contradiction.

Second step. We are going to use Condition (3) in order to 'integrate' the above tessellation of a topological neighbourhood of P into a tessellation of a metric neighbourhood of P.

Fix some $\vartheta < d/2$, where d is provided by IV, and fix some $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} \min\limits_{s \cap s' \neq \varnothing} \varepsilon(s,s',\vartheta/2)$ and $\varepsilon < \vartheta/2$, where $\varepsilon(s,s',\vartheta/2)$ is given by Condition (3).

Given an edge e, we put $N_{e,r} := \bigcup\limits_j \big(I_j^{-1}, \widetilde N(I_je,r)\big)$, where the I_j's correspond to the geometric cycle including e as in 3.1. For $s\in S$, define

$$N_{s,r} := (1,\widetilde N(s,r))\bigcup\big(I_{\overline s},\widetilde N(\overline s,r)\big),\qquad W_s := \pi N_{s,\varepsilon}\bigcup_{e\in E_s}\pi N_{e,\vartheta}.$$

• Let us show that $\varphi : W_s\to N(s,\varepsilon)\bigcup\limits_{e\in E_s} N(e,\vartheta)$ is a bijection.

Choose any $e\in E_s$. As above, define $F := \bigcup\limits_j I_j^{-1}s_{j+1}\cap N(e,\vartheta)$, where the I_j's correspond to the geometric cycle including e as in 3.1, and pick a point $x\in e$. Using the tessellation of a small open ball B centred at x, we can choose $q_0\in B$ living in exactly one of the $I_j^{-1}\widetilde N(I_je,\vartheta)$. Clearly, $F\subset\varphi\pi N_{e,\vartheta}$. It follows from the description 3.2 of formal neighbours that $\varphi:\pi N_{e,\vartheta}\to N(e,\vartheta)$ is injective when restricted to F. Let $q\in N(e,\vartheta)\setminus F$. As above, connecting q_0 and q by a smooth oriented curve $\gamma\subset N(e,\vartheta)$ that does not intersect e and is transverse to F, we can see that γ intersects only the prescribed faces because $\vartheta<d$. We conclude that $\varphi:\pi N_{e,\vartheta}\to N(e,\vartheta)$ is surjective and injective. Since $\vartheta<d/2$, the $N(e,\vartheta)$ are disjoint. Therefore, $\varphi:\bigcup\limits_{e\in E_s}\pi N_{e,\vartheta}\to\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is a bijection.

It is easy to see that

$$s\setminus\bigcup_{e\in E_s}N(e,\vartheta)\subset\varphi\Big(W_s\setminus\bigcup_{e\in E_s}\pi N_{e,\vartheta}\Big)\subset N(s,\varepsilon)\setminus\bigcup_{e\in E_s}N(e,\vartheta).$$

The description 3.2 of formal neighbours implies that $\varphi : W_s\setminus\bigcup\limits_{e\in E_s}\pi N_{e,\vartheta}\to N(s,\varepsilon)\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is injective when restricted to $s\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$. Pick a point $q\in N(s,\varepsilon)\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$ such that $q\notin s$. There exist $x\in s$ and an oriented smooth curve $\gamma\subset N(s,\varepsilon)$ of length $\ell(\gamma)<\varepsilon$ that connects x and q.

We claim that γ can intersect ∂P and $\partial I_{\overline s}P$ only along $s\setminus\partial s$. Indeed, γ cannot intersect the faces of P or of $I_{\overline s}P$ that are disjoint from s because $\varepsilon<d$. Let s' be a face of P or $I_{\overline s}P$ that intersects γ and such that $s\cap s'\neq\varnothing$. By Condition (3) and the choice of ε, we have

$$s'\cap\gamma\subset s'\cap N(s,\varepsilon)\subset\bigcup_{s\diamond e\diamond s'}N(e,\vartheta/2)\subset\bigcup_{e\in E_s}N(e,\vartheta/2),$$

which implies $q\in\bigcup\limits_{e\in E_s}N(e,\vartheta)$ because $\varepsilon<\vartheta/2$. A contradiction. The inequality $\varepsilon<\vartheta/2$ implies that γ does not intersect ∂s.

We can assume that γ is transverse to s. Considering the tessellation of a small ball centred at x introduced earlier, we see that γ first enters the interior of P or of $I_{\overline s}P$. When γ intersects $s\setminus\partial s$, it leaves P and enters $I_{\overline s}P$ or *vice-versa.* As above, $\varphi : W_s\setminus\bigcup\limits_{e\in E_s}\pi N_{e,\vartheta}\to N(s,\varepsilon)\setminus\bigcup\limits_{e\in E_s}N(e,\vartheta)$ is surjective and injective.

• Finally, let us show that the open metric neighbourhood $N := \mathring P\bigcup\limits_{s\in S}N(s,\varepsilon)\bigcup\limits_{e\in E_s}N(e,\vartheta)$ of P is tessellated. (Note that $N(P,\varepsilon)\subset N$.) Define $f(x)=\vartheta$ if $x\in e$ for some edge e of P and $f(x)=\varepsilon$ if $x\in s\setminus\partial s$ for some $s\in S$. If $x\in\mathring P$, we take an arbitrary $f(x)>0$ such that $B\big(x,f(x)\big)\subset\mathring P$. It is immediate that $\varphi W_{P,f}\subset N$ and that $W_{P,f}=[\mathring P]\bigcup\limits_{s\in S}W_s$. Hence, $\varphi:W_{P,f}\to N$ is surjective. If $\varphi w=\varphi w'$, where $w\in[\mathring P]$ and $w'\in W_s$, then $w=[1,x]$, $x\in\mathring P$, and $x\in N(s,\varepsilon)\bigcup\limits_{e\in E_s}N(e,\vartheta)$, implying $w \in W_s$. If $x := \varphi w = \varphi w'$, where $w \in W_s$ and $w' \in W_{s'}$, then $s \neq s'$ and we have two cases: $s \cap s' = \varnothing$ and $s \cap s' \neq \varnothing$. The first case is impossible because $N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta) \subset N(s,\vartheta)$, $N(s',\varepsilon) \bigcup\limits_{e\in E_{s'}} N(e,\vartheta) \subset N(s',\vartheta)$, and $N(s,\vartheta) \cap N(s',\vartheta) = \varnothing$ due to $\vartheta < d/2$.

In the second case, suppose that $x \in N(e_0,\vartheta)$ for some $e_0 \in E_s$. Then $w \in \pi N_{e_0,\vartheta}$ because the bijection $\varphi : \pi N_{e_0,\vartheta} \to N(e_0,\vartheta)$ is a restriction of $\varphi : W_s \to N(s,\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta)$ which is already known to be a bijection. Using $\varepsilon < \vartheta < d/2$ and IV, we can see that the inclusion $x \in N(s',\varepsilon) \bigcup\limits_{e\in E_s} N(e,\vartheta)$ implies $e_0 \in E_{s'}$. So, $w' \in \pi N_{e_0,\vartheta}$ and $w = w'$. The same arguments work if $x \in N(e_0,\vartheta)$ for some $e_0 \in E_{s'}$.

Therefore, we can assume that $x \in N(s,\varepsilon) \cap N(s',\varepsilon)$ and $x \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta)$. We can find some $p \in s' \cap B(x,\varepsilon) \subset s' \cap N(s,2\varepsilon)$. It follows from $p \in B(x,\varepsilon)$, $x \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta)$, and $\varepsilon < \vartheta/2$ that $p \notin \bigcup\limits_{s\diamond e\diamond s'} N(e,\vartheta/2)$. This contradicts the choice of ε and ϑ and completes the proof ■

4. Final Remarks

4.1. Probably, the first version of PPT where the restriction of the combinatorial cycle isometry to an edge is not supposed to be the identity can be found in [Kui, Subsection 3.1, p. 60], although in an implicit form and in the specific case of real hyperbolic 4-space. (It is related to Remark 3.4.) Our version seems to be the first dealing with an angle condition in the situation of nonconstant angle along a common edge of two faces.

4.2. Condition (3) in Theorem 3.5 is not trivial to check in the case of nonconstant curvature. In complex hyperbolic geometry, even in so simple a case as that of bisectors intersecting transversally at a common slice, the proof of this condition requires some analytic effort [AGG, Lemma 2.2.3].

4.3. An important generalization of Theorem 3.5 would be of course a version of PPT for polyhedra admitting faces of codimension > 2. Elaborating explicit conditions that express a good behaviour of faces of all codimensions seems to be the most difficult task here. Indeed, for simplicity, let us assume M to be 3-dimensional. Take a vertex p of P and a small sphere S centred at p. We have an infinitesimal tessellation around generic points in edges, which provides an infinitesimal tessellation of S around its intersections with the edges containing p. Due to the good behaviour of faces and edges, we obtain a tessellation of S. While shrinking the radius of S, the topological picture of this tessellation remains the same. In this way, we visualize a tessellation of the 3-ball bounded by S as being a cone over the tessellation of S.

In the particular case of a compact polyhedron, the conditions expressing a good behaviour of faces must be drastically simplified. For instance, the tessellation of a topological neighbourhood of the polyhedron already implies in this case Tessellation Condition 2.1. We suggest the following formulation: Let M be a Riemannian manifold and let $P \subset M$ be a simple compact PL-polyhedron equipped with a face-pairing providing a family of geometric cycles of edges that contain every edge of P. If Conditions (1) and (2) of Theorem 3.5 hold, then Tessellation Condition 2.1 is satisfied.

This is just a rough outline of a possible proof; getting the general version in question may require some serious effort. We thank Misha Kapovich for pointing out the reference [Ale] (see also the proof of [Ale, Theorem 2]) where similar ideas are applied to compact polyhedra with totally geodesic faces in constant curvature spaces.

5. References

[AGG] S. Anan′in, C. H. Grossi, N. Gusevskii, *Complex hyperbolic structures on disc bundles over surfaces,* to appear in Int. Math. Res. Not., see also http://arxiv.org/abs/math/0511741.

[Ale] A. D. Alexandrov, *On tiling a space with polyhedra,* in A. D. Alexandrov Selected Works I, Classics of Soviet Mathematics Vol. 4, Gordon and Breach Publishers, 1996. x+322 pp.

[Bea] A. F. Beardon, *The Geometry of Discrete Groups,* GTM **91**, Springer-Verlag, New York, 1983. xii+337 pp.

[EPe] D. B. A. Epstein, C. Petronio, *An exposition of Poincaré's polyhedron theorem,* Enseign. Math. **40** (1994), 113–170.

[Kui] N. H. Kuiper, *Hyperbolic* 4*-manifolds and tessellations,* Inst. Hautes Études Sci. Publ. Math. **68** (1988), 47–76.

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